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# Long-time behaviour of a random inhomogeneous field of weakly nonlinear surface gravity waves

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In this paper we investigate nonlinear interactions of narrowband, Gaussian-random, inhomogeneous wavetrains. Alber studied the stability of a homogeneous wave spectrum as a function of the width $\sigma$ of the spectrum. For vanishing bandwidth the deterministic results of Benjamin & Feir on the instability of a uniform wavetrain were rediscovered whereas a homogeneous wave spectrum was found to be stable if the bandwidth is sufficiently large. Clearly, a threshold for instability is present, and in this paper we intend to study the long-time behaviour of a slightly unstable modulation by means of a multiple-timescale technique. Two interesting cases are found. For small but finite bandwidth $\sigma$ the amplitude of the unstable modulation shows initially an overshoot, followed by an oscillation around the time-asymptotic value of the amplitude. This oscillation damps owing to phase mixing except for vanishing bandwidth because then the well-known Fermi–Pasta–Ulam recurrence is found. For large bandwidth, however, no overshoot is found since the damping is overwhelming. In both cases the instability is quenched because of a broadening of the spectrum.

## 1. Introduction

Starting with investigations of Phillips (1960) and Hasselmann (1962, 1963) there has been much interest in the energy transfer due to four-wave interactions in a nearly homogeneous random sea (Hasselmann *et al.* 1973; Watson & West 1975; Willebrand 1975). Longuet-Higgins (1976) derived the narrowband limit of Hasselmann's equation by starting from the nonlinear Schrödinger equation, describing the evolution of the envelope of a narrowband, weakly nonlinear wavetrain. All this nonlinear energy transfer occurs on a rather long timescale since the rate of change of the action density $n$ is proportional to $n^3$. Hence $(\partial n/\partial t)/n = O(\epsilon^4\omega_0)$, where $\epsilon$ is the wave steepness and $\omega_0$ is a typical frequency of the wave field.

A much faster energy transfer is possible in the presence of spatial inhomogeneities. For an inhomogeneous random sea Watson & West (1975) and Willebrand (1975) obtained lower-order corrections to the transport equation of Hasselmann. Also, Alber (1978) and Alber & Saffman (1978) derived an equation describing the evolution of a random narrowband wavetrain. Just like Longuet-Higgins (1976), their starting point was the nonlinear Schrödinger equation or its equivalent for finite depth. Finally, starting from the full equations of motion, Crawford, Saffman & Yuen (1980), following Zakharov's (1968) approach, obtained a unified equation for the evolution of a random field of deep-water waves which accounts for both the effects of spatial

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inhomogeneity and the energy transfer associated with a homogeneous spectrum. From their analysis it became apparent that spatial inhomogeneities gave rise to a much faster energy transfer ($(\partial n/\partial t)/n = O(\epsilon^2\omega_0)$), though this energy transfer is reversible. The energy transfer associated with a homogeneous sea is, however, irreversible.

It should be emphasized that the assumption of an inhomogeneous wave field makes sense because Alber (1978) showed that a homogeneous spectrum is unstable to long-wavelength perturbations if the width of the spectrum is sufficiently small. For a Gaussian spectrum the instability criterion was $\sigma_\omega/\omega_0 = \epsilon$, where $\sigma_\omega$ is the width in frequency space. Similar results were also found by Crawford *et al.* (1980) for a Lorentzian shape of the spectrum. In the limit of vanishing bandwidth the deterministic results of Benjamin & Feir (1967) on the instability of a uniform wavetrain were recovered.

In this paper we wish to investigate nonlinear interactions in an inhomogeneous wave field and we choose as our starting point the nonlinear transport equation for the envelope spectrum of a narrowband, Gaussian-random, surface wavetrain (Alber 1978). The envelope spectrum is just the Fourier transform of the autocorrelation function of the envelope of the wavetrain and contains all the information of the stochastic wave field we need. The technique used by Alber (1978) to obtain the nonlinear transport equation was applied for the first time by Wigner (1932) in quantum mechanics. Applications of this technique to the field of plasma physics have been made by Tappert (1971) and Hasegawa (1975).

According to the results of Alber (1978), Alber & Saffman (1978) and Crawford *et al.* (1980), the random equivalent of the Benjamin–Feir instability has a threshold for instability, and in this paper we intend to study the long-time behaviour of a slightly unstable modulation. We remark that with long-time we mean long on the timescale $(\epsilon^2\omega_0)^{-1}$, which is, however, still short compared with the timescale $(\epsilon^4\omega_0)^{-1}$ for irreversible processes. Then, application of the multiple-timescale technique gives the Duffing equation with complex coefficients,

$$A\frac{\partial^2}{\partial t^2}\Gamma + B\frac{\partial}{\partial t}\Gamma + C\Gamma + D|\Gamma|^2\Gamma = 0 \quad (t > 0), \tag{1}$$

as the evolution equation for the complex amplitude $\Gamma$ of the slightly unstable modulation.

Two interesting special cases are found. For small but finite bandwidth $\sigma$ the amplitude $\Gamma$ initially shows overshoot followed by an oscillation around its time-asymptotic value. This oscillation is damped owing to phase mixing except in the limit of vanishing bandwidth because then the well-known Fermi–Pasta–Ulam (1940) recurrence (see also Lake *et al.* 1977) is recovered. In the latter limit all the coefficients of (1) are real and $B = 0$, and (1) is then similar to the evolution equation of the Benjamin–Feir instability of a deterministic narrow-band wavetrain (Janssen 1981). On the other hand, for large bandwidth (but small enough such that there still is instability) the damping due to phase mixing becomes overwhelming so that no overshoot is found. In the latter case one may neglect the second derivative in (1), and then the evolution equation for $\Gamma$ is just the well-known Landau equation.

We emphasize that, because of the finite bandwidth of the spectrum, limit-cycle behaviour is found instead of recurrence as in the deterministic Benjamin–Feir instability. An attempt is made to explain this 'dissipative' effect of finite bandwidth in terms of phase mixing of a continuum of normal modes of the linear problem, just like phase mixing of van Kampen (1955) modes of the linear Vlasov–Poisson system from plasma physics explains the Landau (1946) damping of the Langmuir oscillations. In addition, for small bandwidth $\sigma$ the transition from stable to unstable involves two modes (a growing and a damped one) which are closely grouped together in frequency space, thus giving a second-order derivative in (1). This explains the overshoot effect. For large bandwidth, however, the 'distance' in frequency space between the two modes is large so that the coupling between these modes becomes unimportant. Although we have not made a detailed comparison of this theory with observations at sea, it is tempting to relate our result to the well-known 'overshoot phenomenon' observed in growing surface gravity waves under the action of the wind (Phillips 1977). This is because we not only have found overshoot in the amplitude of an unstable modulation of a homogeneous spectrum but also because the timescale for nonlinear energy transfer due to inhomogeneities seems to be of the right order of magnitude.

The plan of this paper is as follows. In §2 we derive the equation for the envelope spectrum from the one-dimensional nonlinear Schrödinger equation and we briefly review the linear stability theory of a homogeneous spectrum of random, narrowband wavetrains. In addition, the behaviour of the dispersion relation near the threshold for instability is investigated and in §3 we derive the nonlinear evolution equation for the amplitude of a slightly unstable modulation by means of the multiple-timescale method. We also present a simple interpretation of the saturation of the instability for the case that second harmonics may be neglected (quasilinear approximation) and we conclude with a summary of conclusions (§4). In the appendix we explain the stabilizing effect of finite bandwidth in terms of phase mixing of the normal modes of the linearized equation for the envelope spectrum. We finally remark that we shall not give the details of our calculations because they may be found in a paper of Simon & Rosenbluth (1976), who dealt with similar problems when discussing the saturation of a single mode of the so-called bump-on-tail instability of plasma physics. In particular, this reference should be consulted for a procedure to deal with integrals involving products of generalized functions.

## 2. The transport equation for the envelope spectrum and linear theory

In order to investigate the effect of inhomogeneities such as wave groups on the nonlinear energy transfer of weakly nonlinear water waves we propose to study the nonlinear Schrödinger equation. This equation may be applied to the case of water waves with a narrowband spectrum and small wave steepness so that in a good approximation the surface elevation is given by

$$\zeta \approx \mathrm{Re}\,(A(x,t)\exp \mathrm{i}(k_0 x-\omega_0 t)). \tag{2}$$

Here $\omega_0$ and $k_0$ are the angular frequency and the wavenumber of the carrier wave, which obey the deep-water dispersion relation $\omega_0 = (gk_0)^{\frac{1}{2}}$, and $A(x,t)$ is the slowly varying complex envelope of the wave.

Application of the multiple-timescale technique to the exact deep-water equations then gives the following nonlinear Schrödinger equation for $A(x,t)$:

$$\mathrm{i}\left(\frac{\partial}{\partial t}+\omega_0'\frac{\partial}{\partial x}\right)A+\tfrac{1}{2}\omega_0''\frac{\partial^2}{\partial x^2}A-\tfrac{1}{2}\omega_0 k_0^2|A|^2A = 0, \tag{3}$$

where a prime denotes differentiation with respect to $k$. Transforming to a frame moving with the group velocity $\omega_0'$ and introducing dimensionless units $\tilde{t} = \frac{1}{2}\omega_0 t$,

$\tilde{x} = 2k_0 x$ and $\tilde{A} = k_0 A$, the equation for $\tilde{A}$ (which is now just the wave steepness $\epsilon$ for the uniform wavetrain) reads

$$\mathrm{i}\frac{\partial}{\partial t}A - \frac{\partial^2}{\partial x^2}A - |A|^2 A = 0, \tag{4}$$

where we have dropped the tildes. To proceed, we follow Alber (1978) and define the two-point correlation function $\rho(x_1, x_2, t)$ as

$$\rho(x, r, t) = \langle A(x_1, t)\, A^*(x_2, t)\rangle, \tag{5}$$

where the averaged coordinate $x = \frac{1}{2}(x_1 + x_2)$, the separation coordinate $r = x_2 - x_1$ and the angle brackets denote an ensemble average. The asterisks denote the complex conjugate (c.c.). Following Wigner (1932) and Alber (1978), we obtain the transport equation for $\rho(x, r, t)$ by multiplying the equation (4) for $A(x_1, t)$ by $A^*(x_2, t)$, adding the complex-conjugate expression with $x_1$ and $x_2$ interchanged, and averaging. The result is

$$\mathrm{i}\frac{\partial}{\partial t}\rho(x_1, x_2) - \left(\frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial x_2^2}\right)\rho(x_1, x_2) - \langle A^2(x_1)\, A^*(x_1)\, A^*(x_2)\rangle + \langle A^2(x_2)\, A^*(x_2)\, A^*(x_1)\rangle = 0, \tag{6}$$

hence the rate of change of the two-point correlation function is related to the four-point correlation function. Closure is achieved by assuming the quasi-Gaussian approximation, e.g.

$$\langle A^2(x_1)\, A^*(x_1)\, A^*(x_2)\rangle \approx 2\langle A(x_1)\, A^*(x_1)\rangle\, \langle A(x_1)\, A^*(x_2)\rangle. \tag{7}$$

If, in addition, we transform to the averaged coordinate $x$ and the separation coordinate $r$, we obtain from (6)

$$\mathrm{i}\frac{\partial}{\partial t}\rho(x, r) - 2\frac{\partial^2}{\partial x\, \partial r}\rho(x, r) - 2\rho(x, r)\, [\rho(x + \tfrac{1}{2}r, 0) - \rho(x - \tfrac{1}{2}r, 0)] = 0. \tag{8}$$

Finally, we introduce the envelope spectrum $W$ according to

$$\rho(x, r) = \int_{-\infty}^{\infty} \mathrm{d}p\, \mathrm{e}^{-\mathrm{i}pr}\, W(x, p). \tag{9}$$

When the spatial separation $r = 0$ we obtain the mean-square wave steepness

$$\rho(x, 0) = \langle |A(x)|^2\rangle = \int_{-\infty}^{\infty} \mathrm{d}p\, W(x, p). \tag{10}$$

By application of the standard procedures, we obtain from (8) the following transport equation for $W$:

$$\frac{\partial}{\partial t}W + 2p\frac{\partial}{\partial x}W + 4\sin\left(\frac{1}{2}\frac{\partial^2}{\partial p\, \partial x'}\right) W(x, p)\, \rho(x', 0)|_{x' = x} = 0, \tag{11}$$

where

$$\sin\left(\frac{1}{2}\frac{\partial^2}{\partial p\, \partial x'}\right) = \frac{1}{2\mathrm{i}}\sum_{l=0}^{\infty} \frac{\left(\frac{\mathrm{i}}{2}\frac{\partial^2}{\partial p\, \partial x'}\right)^{2l+1}}{(2l+1)!}. \tag{12}$$

Equation (11) describes the evolution of an inhomogeneous ensemble of narrowband weakly nonlinear wavetrains. The timescale for nonlinear energy transfer can easily be estimated from (8) with the result

$$\frac{1}{\rho}\frac{\partial\rho}{\partial t} = O(\rho) = O(\epsilon^2).$$

This energy transfer (owing to spatial inhomogeneities) is, however, reversible, since (11) is invariant under the transformation $t \to -t$, $p \to -p$. If one includes deviations from Gaussian statistics, which are generated because nonlinearity gives rise to correlations between the different components of the envelope spectrum, irreversible changes in the envelope spectrum of water waves occur. Crawford *et al.* (1980) have shown, however, that these irreversible changes (including the nonlinear energy transfer associated with a homogeneous sea; Hasselmann 1962) occur on the much longer timescale $\tau_{\text{irrev}} = O(\epsilon^{-4})$.

We emphasize that the assumption of an inhomogenous ensemble of wavetrains seems useful because Alber (1978) showed that a homogeneous spectrum is unstable to long-wavelength perturbations if the width of the spectrum is sufficiently small. To see this, we perturb the homogeneous envelope spectrum $W_0(p)$ slightly according to

$$W = W_0(p) + W_1(x, p), \quad W_1 \ll W_0. \tag{13}$$

Then, substitution of (13) in (11) and linearization in $W_1$ gives the linear problem

$$\frac{\partial}{\partial t} W_1 + 2p \frac{\partial}{\partial x} W_1 + 4\left[\sin\left(\frac{1}{2}\frac{\partial^2}{\partial p\, \partial x'}\right) W_0(p)\, \rho_1(x')\right]_{x'=x} = 0, \tag{14}$$

where

$$\rho_1 = \int_{-\infty}^{\infty} \mathrm{d}p\, W_1.$$

Let us consider perturbations of the type

$$W_1 = \hat{W}_1(p, t)\, \mathrm{e}^{\mathrm{i}kx}; \tag{15}$$

then as a result we obtain the following integro-differential equation for $\hat{W}_1$:

$$\frac{\partial}{\partial t} \hat{W}_1 + 2\mathrm{i}kp \hat{W}_1 = -2\mathrm{i}k\eta(p) \int_{-\infty}^{\infty} \mathrm{d}p\, \hat{W}_1, \tag{16}$$

where

$$\eta = \frac{1}{k}[W_0(p + \tfrac{1}{2}k) - W_0(p - \tfrac{1}{2}k)].$$

One may now immediately solve (16) by means of the Laplace-transform technique. In an analogous fashion, Landau (1946) treated the similar-looking linearized Vlasov equation of plasma physics in order to obtain the Landau damping of Langmuir oscillations. The result of the application of the Laplace-transform technique is that for large times $\hat{\rho}_1 = \int \mathrm{d}p \hat{W}_1$ consists of a sum of exponentials (in the case of simple roots of (17)):

$$\hat{\rho}_1 = \sum_j a_j \exp(-\mathrm{i}\omega_j t),$$

where $a_j$ is determined by the initial condition and $\omega_j$ is determined by the dispersion relation

$$D(\nu, k) \equiv 1 + \int_{\smile} \mathrm{d}p \frac{\eta}{p - \nu} = 0, \quad \nu \equiv \frac{\omega}{2k}. \tag{17}$$

For positive times the contour of integration is indented below the singularity at $p = \nu$.

Alber (1978) investigated the dispersion relation (17) for a Gaussian spectrum, whereas Crawford *et al.* (1980) considered the Lorentz spectrum

$$W_0 = \frac{\langle A_0^2 \rangle \sigma}{\pi(p^2 + \sigma^2)}, \tag{18}$$

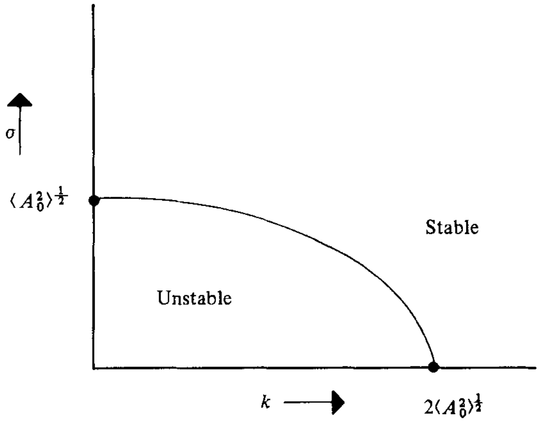


FIGURE 1. Stability diagram of a uniform wavetrain with random phase in one space dimension.

where $\sigma$ is the width of the spectrum and $\langle A_0^2\rangle$ is the mean-square wave steepness. $D$ may be evaluated immediately with the result

$$D = 1+\frac{\langle A_0^2\rangle}{(\nu+\mathrm{i}\sigma)^2-\frac{1}{4}k^2} = 0, \tag{19}$$

so that

$$\nu_{\pm} = -\mathrm{i}\sigma \pm (\tfrac{1}{4}k^2-\langle A_0^2\rangle)^{\frac{1}{2}}. \tag{20}$$

We have instability for $\operatorname{Im}(\nu) > 0$, and the stability boundary consists of the line $k = 0$ and the ellipse $\sigma^2+\frac{1}{4}k^2 = \langle A_0^2\rangle$ (see figure 1).

Note that in the limit of vanishing bandwidth $\sigma$ the growth rate (20) reduces to the result of Benjamin & Feir (1967) for a deterministic wavetrain if one makes the identification $2\langle A_0^2\rangle$ (Gaussian random) $= A_0^2$ (deterministic). We also remark that finite bandwidth $\sigma$ gives a reduction of the growth rate and that when

$$\sigma \geqslant \langle A_0^2\rangle^{\frac{1}{2}}$$

the instability disappears. The question of the nature of the stabilizing effect of finite bandwidth is important. First of all we note that this stabilizing effect is not of a dissipative nature since (20) and also the 'dispersion relation' (17) depend on the sense of time. For positive times the contour of integration (in (17)) has to be indented below the singularity at $p = \nu$, whereas for negative times the contour has to be indented above the singularity. As a result, for negative times the roots of the dispersion relation are just the complex conjugate of the roots for positive times. Therefore our results are symmetrical with respect to the initial point of departure, in agreement with the invariance of the transport equation for the envelope spectrum (11) under time reversal. In the appendix an attempt is made to explain the stabilizing effect of finite bandwidth. To that end we solve the linear problem for $W_1$ (14) by means of the normal-mode approach. In general, for fixed $k$ one then has a discrete and a continuous frequency spectrum. For smooth initial data also the continuum modes are excited, giving rise to phase mixing of the solution to zero (in the case where there is no instability). The only exception is the limit of vanishing bandwidth since then the relevant functions are not smooth. This explanation is entirely analogous to the one given by van Kampen (1955) for the Landau damping of Langmuir waves (see also Case 1959).

In the remaining part of this paper we wish to investigate the long-time behaviour of a single slightly unstable modulation. We therefore consider the initial-value problem for one particular modulation with a fixed wavenumber $k$ near the threshold for instability (see figure 1). In particular, we are concerned with the effect of nonlinearities since they may considerably modify the growth rate of the initially unstable modulation. Before we proceed let us first study the behaviour of the roots of the dispersion relation (17) near the threshold for instability. To this end we choose the averaged wave steepness

$$\langle A_0^2\rangle \equiv \int_{-\infty}^{\infty} \mathrm{d}p\, W_0(p) = \langle A_0^2\rangle_c\,(1+\Delta^2), \quad \Delta^2 \ll 1, \tag{21}$$

where $\langle A_0^2\rangle_c$ is the critical value for which the last unstable mode has just reached the real axis. This critical value of $\langle A_0^2\rangle$ exists by virtue of a discrete wavenumber $k$ so that there are only a finite number of unstable modes.

On application of the Plemelj formulae to the dispersion relation (17), it can easily be shown that at marginal stability $\nu_c$ must coincide with a zero of the function $\eta$,

$$\eta_c(\nu_c) = 0, \tag{22$a$}$$

whereas the critical value of $\langle A_0^2\rangle$ is determined by

$$1 + \mathrm{P}\int_{-\infty}^{\infty} \mathrm{d}p \frac{\eta_c(p)}{p-\nu_c} = 0. \tag{22$b$}$$

Here P denotes the principal value and the subscript c denotes evaluation at the critical value of $\langle A_0^2\rangle$. If one increases the mean-square wave steepness $\langle A_0^2\rangle$ above its critical value by a small fractional amount $\Delta^2$ according to (21) (which is equivalent to $\eta = \eta_c(1+\Delta^2)$), one may solve the dispersion relation (17) approximately by expanding $\nu$ in powers of $\Delta$:

$$\nu = \nu_c + \Delta\nu_1 + \Delta^2\nu_2 + \dots. \tag{23}$$

To second order in $\Delta$ we then obtain from the dispersion relation (17)

$$D = D(\nu_c, \eta_c) + \Delta\nu_1 D^{\mathrm{I}} + \Delta^2\nu_2 D^{\mathrm{I}} + \frac{\Delta^2\nu_1^2}{2!} D^{\mathrm{II}} + \Delta^2(D(\nu_c, \nu_c) - 1) = 0, \tag{24}$$

where $D^{\mathrm{I}} = (\partial D/\partial \nu)_{\nu=\nu_c}$, etc., and we see at once that in lowest order $D(\nu_c, \eta_c) = 0$, which just corresponds to (22$a$, $b$). Before we proceed to the next order we remark that at least the following possibilities may occur.

### (i) *Simple-root transition*

When only a simple root is involved in the transition from stable to unstable we have $D^{\mathrm{I}} = O(1)$. Then, one immediately obtains from (24) that $\nu_1 = 0$ and

$$\nu_2 = \frac{1}{D^{\mathrm{I}}}, \tag{25$a$}$$

or, using the definition of $D$ (cf. (17)),

$$\nu_2 = \frac{1}{\displaystyle\int \mathrm{d}p \frac{\eta_c^{\mathrm{I}}}{p-\nu_c}} = \frac{\mathrm{P}\displaystyle\int \mathrm{d}p[\eta_c^{\mathrm{I}}/(p-\nu_c)] - \pi \mathrm{i}\eta_c^{\mathrm{I}}(\nu_c)}{\left\{\mathrm{P}\displaystyle\int \mathrm{d}p[\eta_c^{\mathrm{I}}/(p-\nu_c)]\right\}^2 + \pi^2[\eta_c^{\mathrm{I}}(\nu_c)]^2}. \tag{25$b$}$$

Here the Roman superscript denotes differentiation with respect to $p$. Hence $\nu$ is an analytic function of the small parameter $\Delta^2$, and according to (25$b$) both a frequency shift and growth of the modulation with wavenumber $k$ is present if the mean-square wave steepness is increased above its critical value.

(ii) *Double-root transition*

When a double root is involved in the transition from stable to unstable we have $D^{\mathrm{I}} = O(\Delta)$, or

$$D^{\mathrm{I}} \equiv \Delta \overline{D^{\mathrm{I}}}, \tag{26}$$

where $\overline{D^{\mathrm{I}}} = O(1)$. Clearly (25$a$) is not meaningful in this case. Returning to (24) we obtain to second order in $\Delta$

$$\tfrac{1}{2}D^{\mathrm{II}}\nu_1^2 + \nu_1 \overline{D^{\mathrm{I}}} - 1 = 0; \tag{27}$$

hence

$$\nu_1 = \frac{-\overline{D^{\mathrm{I}}} \pm \{(\overline{D^{\mathrm{I}}})^2 + 2D^{\mathrm{II}}\}^{\frac{1}{2}}}{D^{\mathrm{II}}}. \tag{28}$$

For real $D^{\mathrm{II}}$ we deal with two complex-conjugate modes if $\overline{D^{\mathrm{I}}} = 0$, whereas finite $\overline{D^{\mathrm{I}}}$ gives an asymmetry in the complex $\nu$-plane. It is also clear that the growth rate of the modes is now of the order $\Delta$ instead of $\Delta^2$. This type of 'singular' behaviour of the growth rate as a function of the small parameter $\Delta^2$ has been found in connection with other instabilities as well. We mention the Kelvin–Helmholtz instability with surface tension included (Drazin 1970; Nayfeh & Saric 1972) and the Benjamin-Feir instability in one space dimension (Janssen 1981).

Let us apply the foregoing considerations to the special case of a Lorentzian profile (18). From (20) the critical value of the mean-square wave steepness is found to be

$$\langle A_0^2\rangle_{\mathrm{c}} = \sigma^2 + \tfrac{1}{4}k^2,$$

and the critical value of $\nu$ is given by $\nu_{\mathrm{c}} = 0$. Assuming first that $\sigma^2 = O(k^2)$, we obtain for $D^{\mathrm{I}}$, using (19),

$$D^{\mathrm{I}} = \frac{-2\mathrm{i}\sigma}{\sigma^2 + \frac{1}{4}k^2} = O\left(\frac{1}{k}\right),$$

so that we are dealing with a simple-root case. A straightforward application of (25$a$) then gives for the first mode (cf. (20) with the plus sign)

$$\nu_+ = \frac{\mathrm{i}\Delta^2}{2\sigma}(\sigma^2 + \tfrac{1}{4}k^2); \tag{29$a$}$$

hence, if the mean-square wave steepness $\langle A_0^2\rangle$ is increased by a small fractional amount above its critical value, one mode moves into the unstable region, according to (29$a$). The other mode, which is damped (cf. (20)) does not experience such a dramatic change. Substitution of (21) in (20) and expanding gives to second order in $\Delta^2$

$$\nu_- = -2\mathrm{i}\sigma - \frac{\mathrm{i}\Delta^2}{2\sigma}(\sigma^2 + \tfrac{1}{4}k^2); \tag{29$b$}$$

hence the damping rate of this mode hardly changes. We have illustrated this in figure 2.

We emphasize that in this case the two modes are widely separated in the complex $\nu$-plane, so that for the long-time behaviour only the $\nu_1$ mode is of importance. Also note that there is no real part of $\nu$ because the spectrum $W_0$ is symmetrical with respect to $p = 0$ (Alber 1978).

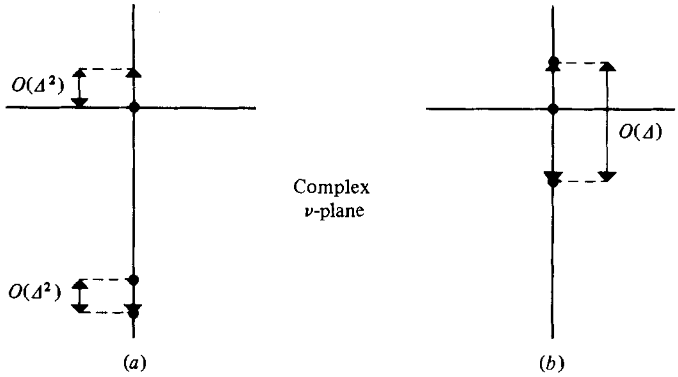


FIGURE 2. Behaviour of the roots of the dispersion relation with a Lorentzian profile near the threshold for instability: (a) broad spectrum, $\sigma = O(k)$; (b) narrow spectrum, $\sigma = O(\Delta)$.

On the other hand, for a narrow spectrum, $\sigma = \Delta\bar{\sigma}$ (see figure 2) we have

$$D^{\mathrm{I}} = \frac{-8\mathrm{i}\Delta\bar{\sigma}}{k^2} + O(\Delta^2);$$

hence we have a double-root case since $D^{\mathrm{I}} = O(\Delta)$. Clearly (29a) is not valid. Application of (28) then gives

$$\nu_{\pm} = -\Delta\mathrm{i}\bar{\sigma}\left[1 \pm \left(1 + \frac{1}{4}\frac{k^2}{\bar{\sigma}^2}\right)^{\frac{1}{2}}\right]. \tag{30}$$

In this case one may therefore expect a different time behaviour compared with a broad spectrum, because for a narrow spectrum the two modes are closely grouped in the complex $\nu$-plane (see figure 2), therefore a strong interaction between these modes is possible.

Hence, depending on the width of the spectrum, or more generally on the order of magnitude of $D^{\mathrm{I}}$, we expect a different long-time behaviour of a single slightly unstable mode. This problem will be studied in §3 by means of the multiple-scale technique. In addition, the effect of nonlinearity is considered.

We finally remark that if one formally takes the limit $D^{\mathrm{II}} \to 0$ in the expression for $\nu_1$ in the double-root case (see (28)) one recovers the simple-root transition as well. This is most easily seen by expanding the square root in (28) for small $D^{\mathrm{II}}$ and taking the limit $D^{\mathrm{II}} \to 0$ afterwards. This simplifies things considerably for the determination of the long-time behaviour of a single slightly unstable mode because we only need to investigate the transition of a double root in detail. The single-root case is then rediscovered by taking the limit $D^{\mathrm{II}} \to 0$.

## 3. Nonlinear evolution of a slightly unstable mode

We wish to obtain the nonlinear evolution of the slightly unstable modes of the random version of the Benjamin–Feir instability. Details of the calculations are given for the double-root transition, whereas the single-root case is rediscovered by taking the appropriate limits. The transition from stable to unstable is obtained by increasing the mean-square wave steepness $\langle A_0^2\rangle$ by a small fractional amount above its critical value. Since $\langle A_0^2\rangle = \int \mathrm{d}p\, W_0$ we take

$$W_0 = W_{0\mathrm{c}}(1+\Delta^2), \quad \Delta^2 \ll 1, \tag{31}$$

where $W_{0c}$ is the 'equilibrium' spectrum such that the modes with the smallest wavenumber are just marginally stable. All other modes are far away from the threshold for instability and they remain stable if $W_0$ is increased by a small fractional amount above its critical value $W_{0c}$.

Let us introduce

$$W = W_0 + \tilde{W}; \tag{32}$$

then (11) becomes

$$\frac{\partial}{\partial t}\tilde{W} + 2p\frac{\partial}{\partial x}\tilde{W} + 4\left[\sin\left(\frac{1}{2}\frac{\partial^2}{\partial p\,\partial x'}\right)(W_{0c}\tilde{\rho}(x') + \tilde{W}\tilde{\rho}(x'))\right]_{x'=x}$$
$$= -4\Delta^2\left[\sin\left(\frac{1}{2}\frac{\partial^2}{\partial p\,\partial x'}\right)W_{0c}\tilde{\rho}(x')\right]_{x'=x}, \tag{33}$$

where we have used (31) and $\tilde{\rho} = \int dp\tilde{W}$. This shows that for finite $\Delta$ the right-hand side of (33) will push the marginally stable modes into the unstable region. This equation is solved by means of the multiple-scale technique. To this end we introduce many timescales $\tau_l = \Delta^l t$; hence

$$\frac{\partial}{\partial t} = \sum_{l=0} \Delta^l \frac{\partial}{\partial \tau_l}, \tag{34}$$

since, according to the analysis of §2, $\tau_0 = t$, $\tau_1 = O(\Delta t)$ for the double-root transition. In addition, we expand $\tilde{W}$ and $\tilde{\rho}$ in powers of $\Delta$,

$$\tilde{W} = \sum_{l=1} \Delta^l W_l, \quad \tilde{\rho} = \sum_{l=1} \Delta^l \rho_l, \tag{35}$$

a usual assumption in asymptotic theories. The coefficients of expansion $W_l$ and $\rho_l$ are functions of all $\tau_l$. Of course, $W_l$ is also a function of $x$ and $p$, and $\rho_l$ is also a function of $x$. Substitution of the expansions (34) and (35) in (33) results in the hierarchy

$$\Delta^l\colon \mathrm{L}W_l = S_l \quad (l = 1, 2, 3, \ldots), \tag{36}$$

where

$$\mathrm{L}W_l \equiv \left\{\frac{\partial}{\partial \tau_0} + 2p\frac{\partial}{\partial x'} + 4\sin\left(\frac{1}{2}\frac{\partial^2}{\partial p\,\partial x'}\right)W_{0c}\int_{-\infty}^{\infty} dp\right\} W_l(p, x')|_{x'=x},$$

and the source terms $S_l$ contain only lower-order $W_m$ with $m \leqslant l-1$. $S_l$ will generate higher harmonics and may also contain terms which give rise to a secular behaviour of $W_l$ on the timescale $\tau_0$. Since many timescales are introduced, there is sufficient freedom to prevent this secular behaviour on the fast timescale $\tau_0$.

The requirement that secularity be absent may be formulated as follows. It is customary (cf. e.g. Simon 1968; Drazin 1970) to introduce the adjoint of L, denoted as $\mathrm{L}^\dagger$. Let $\chi$ be the adjoint function such that

$$\mathrm{L}^\dagger \chi = 0;$$

then, in order to avoid secularities, e.g. to have bounded solutions on the timescale $\tau_0$, we require that

$$(\chi, S_l) = 0. \tag{37}$$

Since we restrict our attention to periodic solutions in $\tau_0$ and $x$, the inner product $(\phi, \psi)$ is defined as

$$(\phi, \psi) \equiv \int d\tau_0\, dx\, dp\, \phi^* \psi,$$

where the integrations over $\tau_0$ and $x$ extend respectively over one period and one wavelength, and $p$ is integrated over the interval $(-\infty, \infty)$. An asterisk denotes the complex conjugate (c.c.).

We remark, however, that for adjoint functions of the form $\chi = \hat{\chi} \exp(\mathrm{i}\theta)$, where $\theta = kx - \omega_c \tau_0$ $(\omega_c = 2k\nu_c)$, the solution of the adjoint problem is given by $\hat{\chi} = 1/(p-\nu_c)$. Therefore the evaluation of the integral over $p$ presents special problems because of the singularity of $\chi$ at $p = \nu_c$. Just as in the linear theory of §2, we resolve this indeterminacy by solving the hierarchy of equations (36) by means of the Laplace-transform technique. Hence, writing

$$S_l = s_{l,0} + s_{l,1}\,\mathrm{e}^{\mathrm{i}\theta} + \text{higher harmonics} + \text{c.c.}$$

for the source term, we obtain

$$\mathrm{L}W_l = s_{l,0} + s_{l,1}\,\mathrm{e}^{\mathrm{i}\theta} + \text{higher harmonics} + \text{c.c.} \tag{38}$$

Both the terms $s_{l,0}$ and $s_{l,1} \exp(\mathrm{i}\theta)$ give rise to secularity. Solving (38) by means of the Laplace-transform technique, absence of secularity is found if

$$s_{l,0} = 0, \tag{39a}$$

whereas the solvability condition for the term oscillating with eigenfrequency $\omega_c$ reads

$$\int_{\rightharpoondown} \mathrm{d}p \frac{s_{l,1}}{p-\nu_c} = 0 \quad (t > 0). \tag{39b}$$

Just as in the linear theory of §2, for positive times the integration contour is indented below the singularity at $p = \nu_c$, whereas for negative times it is indented above the singularity. Note that the solvability condition (39*b*) is of the form (37), except that the Laplace-transform technique gives a description of how to deal with the singularity at $p = \nu_c$, thereby resolving the indeterminacy. For more details see Simon & Rosenbluth (1976).

In order to obtain a unique solution of the hierarchy (37), initial and boundary conditions have to be specified. At $t = 0$, we assume that only the unstable mode(s) and its higher harmonics are excited. In addition, we require periodic boundary conditions in $x$-space, and that $W_l$ vanishes sufficiently rapidly for $p \to \pm\infty$ so that all the integrals over $p$ exist.

In §§3.1–3.4 the hierarchy of equations (37) is solved order by order, subject to the initial and boundary conditions. Secularities can be avoided by application of the conditions (39*a*, *b*), and as a result we obtain a nonlinear evolution equation for the amplitude of the slightly unstable modulation. Both the effect of the modification of the equilibrium $W_{0c}$ and second harmonics stabilize the linearly unstable modulation. In the so-called quasilinear approximation (i.e. the effect of second harmonics is neglected) a simple interpretation of the saturation mechanism is given. Also the transition of a single root is discussed.

### 3.1. *First-order theory* (*double root*)

In first order the linear problem that has been investigated in §2 results, and because of the particular choice of the spectrum (namely $W_0 = W_{0c}$) the two modes with the smallest possible wavenumber $k$ are just stable. In view of the initial conditions the solution reads

$$W_1 = \frac{-\Gamma\eta_c(p)}{p-\nu_c}\,\mathrm{e}^{\mathrm{i}\theta} + \text{c.c.}, \tag{40}$$

where $\theta = kx - \omega_c \tau_0$ and $\Gamma$ is a complex amplitude which is still an unknown function of the timescales $\tau_1, \tau_2, \ldots$. One can easily check that (40) is a solution of the linear problem (14) (with $W_0 = W_{0c}$) provided that the conditions (22*a*, *b*) are satisfied. We incidentally remark that by virtue of (22*a*), i.e. $\eta_c(\nu_c) = 0$, $W_1$ is a well-defined function of $p$. For a double root in the dispersion relation a solution proportional to $\tau_0 \exp(\mathrm{i}\theta)$ is also found. We do not need this solution as a starting point because it is automatically generated by the perturbation scheme.

### 3.2. *Second-order theory* (*double root*)

In second order we obtain

$$\mathrm{L} W_2 = -\frac{\partial}{\partial \tau_1} W_1 - 4\left[\sin\left(\frac{1}{2}\frac{\partial^2}{\partial p\, \partial x'}\right) W_1 \rho_1(x')\right]_{x'=x}, \tag{41}$$

where $\rho_1 = \Gamma \exp(\mathrm{i}\theta) + \text{c.c.}$ The first term on the right-hand side may produce secularity since it oscillates with frequency $\omega_c$. Application of the solvability condition (39*b*) gives

$$\int_{\smile\!\!>} \mathrm{d}p \frac{\eta_c}{(p-\nu_c)^2} = D^{\mathrm{I}} = 0. \tag{42}$$

Since for the double-root transition $D^{\mathrm{I}} = O(\Delta)$ by assumption, the solvability condition (42) is satisfied to the required order. Elimination of $W_1$ and $\rho_1$ from (41) results in

$$\mathrm{L} W_2 = \frac{\eta_c}{p-\nu_c}\frac{\partial}{\partial \tau_1}\Gamma \mathrm{e}^{\mathrm{i}\theta} + 4\mathrm{i}\Gamma^2 \sinh\left(\frac{k}{2}\frac{\partial}{\partial p}\right)\left(\frac{\eta_c}{p-\nu_c}\right)\mathrm{e}^{2\mathrm{i}\theta} + \text{c.c}, \tag{43}$$

where we have introduced the notation

$$2\sinh\left(\frac{k}{2}\frac{\partial}{\partial p}\right) f = f(p+\tfrac{1}{2}k) - f(p-\tfrac{1}{2}k).$$

The solution of (43) reads

$$W_2 = W_{20}(p, \tau_1) + (W_{21}\mathrm{e}^{\mathrm{i}\theta} + W_{22}\mathrm{e}^{2\mathrm{i}\theta} + \text{c.c.}), \tag{44}$$

where

$$W_{21} = \frac{1}{2\mathrm{i}k}\frac{\eta_c}{(p-\nu_c)^2}\frac{\partial}{\partial \tau_1}\Gamma,$$

$$W_{22} = \frac{\Gamma^2}{k(p-\nu_c)}\left\{\sinh\left(\frac{k}{2}\frac{\partial}{\partial p}\right)\left(\frac{\eta_c}{p-\nu_c}\right) - \frac{\eta_c(p,2k)}{D(\nu_c,2k)}\int_{\smile\!\!>}\frac{\mathrm{d}p}{p-\nu_c}\sinh\left(\frac{k}{2}\frac{\partial}{\partial p}\right)\left(\frac{\eta_c}{p-\nu_c}\right)\right\}.$$

Here $\eta_c(p, 2k)$ and $D(\nu_c, 2k)$ are evaluated at $2k$. In second order the term giving modification of the equilibrium, $W_{20}$, is still undetermined, but will be in third order. We finally make the following remark regarding the solution of $W_{21}$. The relevant equation reads

$$(p-\nu_c) W_{21} + \eta_c \int \mathrm{d}p\, W_{21} = \frac{1}{2\mathrm{i}k}\frac{\eta_c}{p-\nu_c}\frac{\partial}{\partial \tau_1}\Gamma, \tag{45}$$

and the solution in (44) is obtained by neglecting the integral $\int \mathrm{d}p\, W_{21}$, in agreement with our assumption

$$D^{\mathrm{I}} = \int_{\smile\!\!>} \frac{\eta_c}{(p-\nu_c)^2}\mathrm{d}p = (\Delta).$$

Of course, although the effect of the integral $\int \mathrm{d}p\, W_{21}$ is neglected in second order, we have to retain it in third order.

### 3.3. *Third-order theory* (*double-root*)

The third-order problem reads

$$\mathrm{L}W_3 = S_3 = -\frac{\partial}{\partial\tau_2}W_1 - \frac{\partial}{\partial\tau_1}W_2 - \frac{2\mathrm{i}k\eta_\mathrm{c}}{\varDelta}\int \mathrm{d}p\,(W_{21}\,\mathrm{e}^{\mathrm{i}\theta}+\mathrm{c.c.}) - 4\left[\sin\left(\frac{1}{2}\frac{\partial^2}{\partial p\,\partial x'}\right)\{\rho_1(x')\,(W_{0\mathrm{c}}+W_2)+\rho_2(x')\,W_1\}\right]_{x'=x}. \quad (46)$$

The source term $S_3$ contains a steady-state term and a term oscillating with frequency $\omega_\mathrm{c}$, which, as already noted, both produce secularity. Application of condition (39*a*) then gives an equation for the modification of the equilibrium,

$$\frac{\partial}{\partial\tau_1}W_{20} = \frac{2}{k}\frac{\partial}{\partial\tau_1}|\Gamma|^2\sinh\left(\frac{k}{2}\frac{\partial}{\partial p}\right)\left\{\frac{\eta_\mathrm{c}}{(p-\nu_\mathrm{c})^2}\right\},$$

which may immediately be integrated with respect to $\tau_1$,

$$W_{20} = \frac{2}{k}|\Gamma|^2\sinh\left(\frac{k}{2}\frac{\partial}{\partial p}\right)\left\{\frac{\eta_\mathrm{c}}{(p-\nu_\mathrm{c})^2}\right\}, \quad (47)$$

for a particular choice of the initial condition on $W_{20}$. We remark that $W_{20}$ is a singular function leading to non-summable integrals on the real axis when multiplied by a 'good' function. In order to regularize this function we follow Gelfand & Shilov (1964) and Simon & Rosenbluth (1976) and add a functional to (47) concentrated at $p+\frac{1}{2}k = \nu_\mathrm{c}$ and $p-\frac{1}{2}k = \nu_\mathrm{c}$, i.e.

$$W_{20} = \frac{2}{k}|\Gamma|^2\sinh\left(\frac{k}{2}\frac{\partial}{\partial p}\right)\left\{\frac{\eta_\mathrm{c}}{(p-\nu_\mathrm{c})^2} + \sum_{n=0}^{\infty}\alpha_n\,\delta^{(n)}(p-\nu_\mathrm{c})\right\}, \quad (48)$$

where $\delta^{(n)}$ is the $n$th derivative of the delta function and $(p-\nu_\mathrm{c})^{-2}$ is the usual generalized function. Since $\alpha_n$ is arbitrary we may generate all regularizations of $W_{20}$. The only restrictions come from the fact that all solutions of our basic equation (11) satisfy certain conservation laws. The first few are given by

$$\frac{\mathrm{d}}{\mathrm{d}t}\int \mathrm{d}x\,\mathrm{d}p\,W = 0, \quad \frac{\mathrm{d}}{\mathrm{d}t}\int \mathrm{d}x\,\mathrm{d}p\,pW = 0, \quad \frac{\mathrm{d}}{\mathrm{d}t}\left[\int \mathrm{d}x\,\mathrm{d}p\,p^2W - \int \mathrm{d}x\,\rho^2\right] = 0, \quad (49a, b, c)$$

assuming periodic boundary conditions in $x$-space and the vanishing of $W$ for large $p$. Using the expansion (35), to second order in $\varDelta$ (49*a*) does not impose a restriction on the coefficients $\alpha_n$ because

$$\int \mathrm{d}p\,W_{20} = 0.$$

From (49*b*) we find at once

$$\alpha_0 = -\mathrm{P}\int \mathrm{d}p\frac{\eta_\mathrm{c}}{(p-\nu_\mathrm{c})^2},$$

and from (49*c*) we obtain to second order in $\varDelta$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\int \mathrm{d}x\,\mathrm{d}p\,p^2W_{20} - \int \mathrm{d}x\,\rho_1^2\right] = 0, \quad (50)$$

since the integral over $W_{0\mathrm{c}}$ is constant in time. One immediately finds $\alpha_1 = 0$. In this fashion all the conservation laws may be satisfied.

We incidentally note that the conservation laws (49) admit a simple physical interpretation. Equation (49*a*) states that the averaged energy is conserved, whereas (49*b*) tells us that the averaged wavenumber $\langle p\rangle$ defined as

$$\langle p\rangle \equiv \frac{\int \mathrm{d}x\,\mathrm{d}p\,p W}{\int \mathrm{d}x\,\mathrm{d}p\,W}$$

is conserved. Finally (49*c*) provides us with information on the evolution of the width $\sigma$ of the spectrum, defined as

$$\sigma^2 \equiv \frac{\int \mathrm{d}x\,\mathrm{d}p\,p^2 W}{\int \mathrm{d}x\,\mathrm{d}p\,W}.$$

In particular, the width of the spectrum increases in the presence of an unstable modulation.

As already remarked, the source term $S_3$ also contains a term oscillating with the eigenfrequency $\omega_c$ of the linear system and thus may result in secularity in $W_3$. Then application of the solvability condition (39*b*) results in the following Duffing equation (with complex coefficients) for the complex amplitude $\Gamma$ of the unstable modulation:

$$A\frac{\partial^2}{\partial\tau_1^2}\Gamma+\frac{B}{\Delta}\frac{\partial}{\partial\tau_1}\Gamma+C\Gamma+D|\Gamma|^2\Gamma=0 \quad (t>0), \tag{51a}$$

where

$$\left.\begin{aligned} A &= D^{\mathrm{II}} = 2\int_{\smile}\frac{\eta_c}{(p-\nu_c)^3}\mathrm{d}p,\\ B &= -4\mathrm{i}kD^{\mathrm{I}} = -4\mathrm{i}k\int_{\smile}\frac{\eta_c}{(p-\nu_c)^2}\mathrm{d}p,\\ C &= 8k^2,\\ D &= -16k\int_{\smile}\frac{\mathrm{d}p}{p-\nu_c}\left[\sinh\left(\frac{k}{2}\frac{\partial}{\partial p}\right)(\hat{W}_{20}-\hat{W}_{22})+\hat{\rho}_{22}\sinh\left(k\frac{\partial}{\partial p}\right)\hat{W}_1\right].\end{aligned}\right\} \tag{51b}$$

Here $\hat{W}_{20}=W_{20}/|\Gamma|^2$, $\hat{W}_{22}=W_{22}/\Gamma^2$, $\hat{\rho}_{22}=\int\mathrm{d}p\,\hat{W}_{22}$ and $\hat{W}_1=W_1/\Gamma$. We emphasize again that our result depends on the sense of time. For positive times, the contour of integration is indented below the singularities on the real $p$-axis, whereas for negative times the contour is indented above the singularities. As a result, for positive and negative times the same time behaviour is found from (51*a*).

We remark that nonlinear effects considerably modify the growth rate of the linearly unstable modulation. Let us comment on the so-called quasilinear approximation (i.e. the effect of second harmonics is neglected). From the conservation laws (49*c*) or (50) we see at once that the modification of the equilibrium $\Delta^2 W_{20}$ gives rise to a broadening of the final time distribution function $W_0+\Delta^2 W_{20}$. Since, as already noted, the effect of finite bandwidth is stabilizing, it is plausible that the modification of the equilibrium quenches the instability. Thus, the effect of the unstable modulation on the equilibrium is such as to increase the width of the spectrum $W_0+\Delta^2 W_{20}$, thereby making the new equilibrium neutrally stable.

In the quasilinear case there is therefore a relatively simple interpretation of the saturation of the instability. One can obtain it by requiring that the final time spectrum $W_0+\Delta^2 W_{20}$ be such as to make the new system neutrally stable. We do not show this straightforward calculation here, but the reader can easily check this by substituting $W_0+\Delta^2 W_{20}$ into the dispersion relation (17) and imposing the condition of neutral stability. In the full nonlinear case such a simple interpretation is not possible.

The expression for $D$, measuring the effect of nonlinearity, may be simplified considerably. To that end we use the expressions for $\hat{W}_1$, $\hat{W}_{20}$, $\hat{W}_{22}$ and $\hat{\rho}_{22}$ and we realize that the sinh operator is given by a similar series as the sin operator in (12). Partial integration and some simplifications then give the result

$$D = 16\int_{\rightarrow} \frac{\mathrm{d}p}{(p-\nu_c)^3}\eta(p). \tag{51c}$$

We emphasize that the nonlinear evolution equation (51*a*) holds for general 'equilibria' $W_0$ provided that at the threshold there is a double root. For the special case of a Lorentz profile with a small bandwidth $\sigma = \Delta\bar{\sigma}$ all the integrals in (51*c*) may be evaluated. To lowest order in $\sigma$ the result is (full nonlinear)

$$\frac{\partial^2}{\partial\tau_1^2}\Gamma+4\bar{\sigma}k\frac{\partial}{\partial\tau_1}\Gamma-k^4\Gamma+8|\Gamma|^2\Gamma = 0. \tag{52}$$

Remark that in the linear approximation we obtain two modes, namely a growing one and a damped one (cf. (29*a*, *b*)). Also, in the case of a spectrum with vanishing bandwidth, i.e. $\bar{\sigma}=0$, we rediscover the well-known Fermi–Pasta–Ulam (1940) recurrence, since (52) with $\bar{\sigma}=0$ gives periodic solutions in time (see figure 3). This is very reassuring since also the long-time behaviour of a single, slightly unstable modulation of a deterministic, uniform wavetrain exhibits the Fermi–Pasta–Ulam recurrence phenomenon (Yuen & Ferguson 1978; Janssen 1981; Infeld 1981).

Also, the saturation level of the instability is in agreement with the deterministic results.

For finite bandwidth, however, the recurrence is not perfect, because of the dissipative term in (52). Let us study this case in more detail. Since $\Gamma$ is complex, we put

$$\Gamma = \rho\,\mathrm{e}^{\mathrm{i}\phi}$$

with real $\rho$ and $\phi$ to obtain from (52) the following coupled set of equations:

$$\frac{\partial^2}{\partial\tau_1^2}\rho-\rho\left(\frac{\partial\phi}{\partial\tau_1}\right)^2+\alpha^2\frac{\partial}{\partial\tau_1}\rho-\gamma^2\rho+\beta^2\rho^3 = 0, \tag{53a}$$

$$\rho\frac{\partial^2}{\partial\tau_1^2}\phi+2\frac{\partial}{\partial\tau_1}\rho\frac{\partial}{\partial\tau_1}\phi+\alpha^2\rho\frac{\partial}{\partial\tau_1}\phi = 0, \tag{53b}$$

where $\alpha^2 = 4\bar{\sigma}k$, $\gamma^2 = k^4$ and $\beta^2 = 8$. Equation (53*b*) may be integrated at once to give

$$\rho^2\frac{\partial}{\partial\tau_1}\phi = a\exp\left(-\alpha^2\tau_1\right),$$

where $a$ is a constant to be determined from the initial condition on the frequency shift $\partial\phi/\partial\tau_1$. Since for the Lorentz profile there is no linear frequency shift (cf. (29*a*)), we take $\partial\phi(0)/\partial\tau_1 = 0$, hence $a=0$. As a result, (53*a*) becomes

$$\frac{\partial^2}{\partial\tau_1^2}\rho+\alpha^2\frac{\partial}{\partial\tau_1}\rho-\gamma^2\rho+\beta^2\rho^3 = 0. \tag{54}$$

This equation has the features sketched in figure 3. Thus, for small but finite bandwidth, the amplitude $\Gamma$ (or $\rho$) shows initially an overshoot followed by a damped

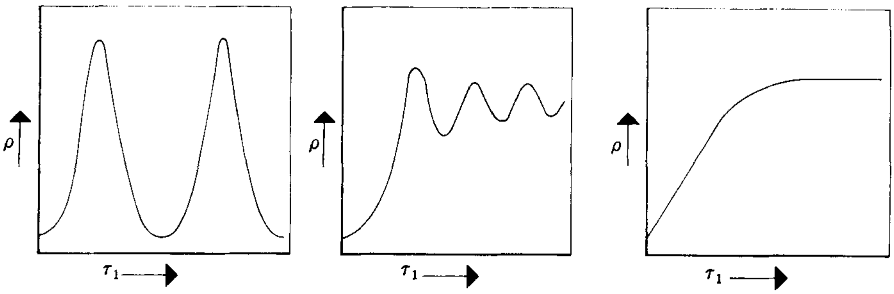


FIGURE 3. Long-time behaviour of the random version of the Benjamin–Feir instability. Shown are the cases of a uniform wavetrain ($\sigma = 0$), a narrow spectrum $\sigma = O(\Delta)$, and a broad spectrum $\sigma = O(1)$.

oscillation around its time-asymptotic value. This damping is caused by phase mixing, as explained in the appendix. Only in the limit of vanishing bandwidth (i.e. $\alpha^2 = 0$) we have perfect recurrence.

### 3.4. *Evolution in time for the simple-root transition*

For the simple-root transition the growth of the unstable mode is of order $\Delta^2$ according to (23) and (25*a*). Therefore, in order to study the long-time behaviour of the random version of the Benjamin–Feir instability of a broad spectrum, one introduces many timescales $\tau_{2l} = \Delta^{2l}t$ $(l = 0, 1, 2, \ldots)$. In combination with the expansions for $\tilde{W}$ and $\tilde{\rho}$ given in (35), one then arrives at a hierarchy of equations of the form given in (36). Subsequently one solves this hierarchy order by order subject to the initial and boundary conditions, while secularities can be avoided by application of the solvability conditions (39*a*, *b*). We have done the details of the calculations, but no purpose is served in reproducing calculations which are entirely analogous to those presented in §§3.1–3.3 (for a similar problem, namely single-mode saturation of the bump-on-tail instability of plasma physics, see Janssen & Rasmussen 1981). We merely note that, just as in the linear theory of §2, the evolution equation for the simple-root case may be obtained from the evolution equation for the double root by formally taking the limit $D^{\mathrm{II}} \to 0$ in (51). Next, introducing the timescale $\tau_2 = \Delta\tau_1$, we obtain for the simple-root case the nonlinear Landau equation (with complex coefficients) for the complex amplitude $\Gamma$ of the unstable modulation:

$$B\frac{\partial}{\partial\tau_2}\Gamma + C\Gamma + D|\Gamma|^2\,\Gamma = 0 \quad (t > 0), \tag{55}$$

where the coefficients $B$, $C$ and $D$ are given in (51*b*). The nonlinear Landau equation can be solved exactly, and by virtue of the fact that it is a first-order differential equation its solution exhibits no overshoot. The reason apparently is that for large bandwidth the damping owing to phase mixing becomes overwhelming so that overshoot is fully suppressed.

Again, (55) holds for broad spectra, and for the special case of a Lorentz profile the coefficients $B$, $C$ and $D$ may be evaluated. The result is

$$\frac{\partial}{\partial\tau_2}\Gamma - \gamma\Gamma + \beta|\Gamma|^2\Gamma = 0, \tag{56}$$

where

$$\gamma = k\langle A_0^2\rangle\,\sigma,$$

$$\beta = \frac{k^2 - 12\sigma^2}{2\sigma k(\sigma^2 + \frac{1}{4}k^2)}.$$

For $k^2 > \frac{1}{12}\sigma^2$, nonlinearity is stabilizing, and in this paper we confine ourselves to this case.

Introducing $\Gamma = \rho\, e^{i\phi}$, we find at once that $\partial\phi/\partial\tau_2 = 0$, so that the equation for the amplitude reads

$$\frac{\partial}{\partial\tau_2}\rho - \gamma\rho + \beta\rho^3 = 0.$$

We have plotted the solution for $\rho$ in figure 3, which shows that there is no overshoot.

To conclude this section we remark that for spectra $W_0(p)$, which are symmetrical with respect to $p = 0$, the coefficients in the evolution equations (51 $a$) and (55) are always real. As a consequence, no frequency shift is found as is evident from the solutions for the double-root transition (in this case $\partial\phi/\partial\tau_1$ decays to zero for large $t$), and for the simple-root transition (here $\partial\phi/\partial\tau_2 = 0$). Only for asymmetrical spectra do the coefficients $A$, $B$, $C$ and $D$ become complex, giving rise to a shift in the oscillation frequency of the unstable modulation.

## 4. Summary of conclusions

In this paper we have studied the long-time behaviour of a random inhomogeneous field of weakly nonlinear wavetrains. According to a linear stability analysis a homogeneous spectrum is unstable to long-wavelength perturbations if the width of the spectrum is sufficiently small. A threshold for instability is present. By increasing the averaged wave steepness by a small fractional amount above its critical value, only one mode moves into the unstable region since periodic boundary conditions are assumed (hence we have discrete wavenumbers). The evolution in time of this slightly unstable mode is determined by means of the multiple-timescale analysis. Two interesting cases are found. For small but finite bandwidth the amplitude of the unstable modulation shows overshoot followed by an oscillation around its time-asymptotic value. This oscillation is damped owing to phase mixing except in the limit of vanishing bandwidth because then there is perfect recurrence. For large bandwidth the damping due to phase mixing becomes overwhelming so that no overshoot is found.

It is tempting to relate our result on the overshoot of sidebands to the well-known 'overshoot phenomenon' observed in growing surface gravity waves under the action of the wind (Phillips 1977). In addition, the timescale $\tau$ (which is given by $\tau^{-1} = \epsilon^2\omega_0$) for this nonlinear process seems to be of the right order of magnitude. Of course, in an honest comparison the effect of the wind input should also be taken into account; this, however, is beyond the scope of the present paper.

Still, the question remains as to whether the random field of surface gravity waves has to be regarded as inhomogeneous or not. The evidence given by Alber (1978), namely that the width of the spectrum of growing sea waves is just of the order of the wave steepness, is very interesting but certainly not conclusive. Perhaps a convincing explanation of the 'overshoot phenomenon' in terms of the nonlinear energy transfer due to inhomogeneities may shed some light on this problem, since the assumption of an inhomogeneous sea seems hard to test experimentally.

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## Appendix

Let us study the stability of a homogeneous spectrum in the framework of a normal mode analysis in order to give an explanation of the stabilizing effect of finite bandwidth. For simplicity, only stable homogeneous spectra are considered. Our starting point is the equation (16) for the perturbation $\hat{W}_1$:

$$\frac{\partial}{\partial t}\hat{W}_1 + 2\mathrm{i}kp\hat{W}_1 = -2\mathrm{i}k\eta(p)\int_{-\infty}^{\infty}\mathrm{d}p\,\hat{W}_1. \tag{A 1}$$

Instead of solving (A 1) by means of the Laplace-transform technique we look for solutions in terms of normal modes:

$$\hat{W}_1 = \tilde{W}_1\exp(-\mathrm{i}\omega t). \tag{A 2}$$

Substitution of (A 2) in (A 1) gives

$$(p-\nu)\,\tilde{W}_1 + \eta\int_{-\infty}^{\infty}\mathrm{d}p\,\tilde{W}_1 = 0, \quad \nu = \frac{\omega}{2k}. \tag{A 3}$$

For a stable equilibrium, $\omega$ and $\nu$ are real. Choosing the normalization

$$\int_{-\infty}^{\infty}\mathrm{d}p\,\tilde{W}_1 = 1, \tag{A 4}$$

the general solution of (A 3) reads

$$\tilde{W}_1 = -\eta(p)\frac{\mathrm{P}}{p-\nu} + \lambda(\nu)\,\delta(p-\nu). \tag{A 5}$$

The symbol P means that the principal value has to be taken when integrating over a domain that includes the point $p = \nu$. From the normalization condition (A 4) we obtain

$$\lambda(\nu) = 1 + \mathrm{P}\int_{-\infty}^{\infty}\frac{\eta(p)}{p-\nu}\mathrm{d}p. \tag{A 6}$$

This condition can be fulfilled for every $\nu$ by choosing $\lambda$ appropriately. Thus the appearance of the parameter $\lambda$ makes it possible to satisfy the normalization condition without having to relate $\nu$ (or $\omega$) to $k$. We therefore have a continuous spectrum.

For special cases, i.e. $\eta(\nu) = 0$, the solution is different from the one given in (A 5). For the present discussion these possibilities are disregarded (see e.g. Case 1959). Also note that for stable equilibria no discrete eigenvalues exist.

The general solution of (A 1) is a superposition of the undamped normal modes (A 5), e.g.

$$\hat{W}_1 = \int_{-\infty}^{\infty}\mathrm{d}\nu\,c(\nu)\,\tilde{W}_1(p,\nu)\,\mathrm{e}^{-2\mathrm{i}\nu kt}, \tag{A 7}$$

where the coefficient $c(\nu)$ is to be determined from the initial condition, i.e.

$$\hat{W}_1(p,0) = \int_{-\infty}^{\infty}\mathrm{d}\nu\,c(\nu)\,\tilde{W}_1(p,\nu). \tag{A 8}$$

Inserting (A 5) into (A 8) we arrive at the following singular integral equation for $c$:

$$\lambda(p)\,c(p) + \mathrm{P}\int_{-\infty}^{\infty}\mathrm{d}\nu\frac{\eta(p)\,c(\nu)}{\nu-p} = \hat{W}_1(p,0), \tag{A 9}$$

which may be solved with the standard techniques from complex analysis with the result

$$c(\nu) = \frac{G^+(\nu)}{D^+(\nu)} - \frac{G^-(\nu)}{D^-(\nu)}, \tag{A 10}$$

where

$$G = \frac{1}{2\pi \mathrm{i}} \int \frac{\mathrm{d}p}{p-z} \hat{W}_1(p,0), \quad D = 1 + \int \mathrm{d}p \frac{\eta(p)}{p-z}, \quad z \,\text{complex},$$

and $G^+(\nu) = \lim_{z \downarrow \nu} G(z)$, etc. In principle we have solved now the initial-value problem for $\hat{W}_1$ by means of the normal mode approach. Let us finally study the time behaviour of one of the moments of $\hat{W}_1$, e.g.

$$\rho_1 = \int \mathrm{d}p \, \hat{W}_1, \tag{A 11}$$

where $\rho_1$ measures the energy of the mode. Using the normalization for $\hat{W}_1$, we have

$$\rho_1 = \int_{-\infty}^{\infty} \mathrm{d}\nu \left[\frac{G^+}{D^+} - \frac{G^-}{D^-}\right] \mathrm{e}^{-2\mathrm{i}k\nu t}. \tag{A 12}$$

We remark that $D^+(z) = D(z)$, $\mathrm{Im}\,(z) > 0$, has by assumption no zeros in the upper half of the complex $z$-plane because it is just the dispersion relation with $\mathrm{Im}\,(\nu) > 0$. Therefore $D^-$ has no zeros in the lower half-plane, and $G^-/D^-$ is an entire function in the lower half of the complex $\nu$-plane. As a result, for positive times the integral involving $G^-/D^-$ in (A 12) vanishes. Hence

$$\rho_1 = \int_{-\infty}^{\infty} \mathrm{d}\nu \frac{G^+(\nu)}{D^+(\nu)} \mathrm{e}^{-2\mathrm{i}k\nu t} \quad (t > 0). \tag{A 13}$$

As is evident from this expression, for smooth initial data, a continuum of normal modes is excited. Since for large times $\exp(-2\mathrm{i}k\nu t)$ will be an erratic function of $\nu$, the integral will vanish for large $t$. Thus moments such as $\rho_1$ decay to zero owing to phase mixing of the continuum. However, the precise details of this decay depend also on the form of $G^+/D^+$.

We emphasize the condition of the smoothness of $G^+/D^+$. If one excites at $t = 0$ a single normal mode (which is highly singular) then $\rho_1$ will be undamped. Also, in the limit of vanishing bandwidth, $G^+/D^+$ is not smooth, since $D^+(\nu)$ vanishes on the real $\nu$-axis. Therefore, in this case the moments of $W_1$ will not decay to zero. For finite bandwidth $D^+(\nu)$ does not vanish on the real axis, so that $G^+/D^+$ is smooth. The result is that for finite bandwidth $\rho_1$ decays to zero, where the decay rate is inversely proportional to the bandwidth $\sigma$.

We finally remark that our conclusion regarding the decay for smooth initial data only holds for the moments of $\hat{W}_1$ and not for $\hat{W}_1$ itself. However, for large times $\hat{W}_1$ will be an erratic function of $p$ such that the moments of $\hat{W}_1$ will decay to zero. A similar explanation was given by van Kampen (1955) for the Landau damping of Langmuir waves.

## REFERENCES


Alber, I. E. 1978 The effects of randomness on the stability of two-dimensional surface wavetrains. *Proc. R. Soc. Lond.* **A363,** 545–546.

Alber, I. E. & Saffman, P. G. 1978 Stability of random nonlinear deep water waves with finite bandwidth spectra. *TRW Defense and Space Systems Group Rep.* no. 31326-6035-RU-00.

Benjamin, T. B. & Feir, J. E. 1967 The disintegration of wave trains on deep water. Part 1. Theory. *J. Fluid Mech.* **27**, 417–430.

Case, K. M. 1959 Plasma oscillations. *Ann. Phys.* (*N.Y.*) **7**, 349.

CRAWFORD, D. R., SAFFMAN, P. G. & YUEN, H. C. 1980 Evolution of a random inhomogeneous field of nonlinear deep-water gravity waves. *Wave Motion* **2**, 1–16.

DRAZIN, P. G. 1970 Kelvin–Helmholtz instability of finite amplitude. *J. Fluid Mech.* **42**, 321.

FERMI, E., PASTA, J. & ULAM, S. 1940 Studies of nonlinear problems. In *Collected papers of Enrico Fermi*, vol. 2, p. 978. University of Chicago Press, 1962.

GELFAND, I. M. & SHILOV, G. E. 1964 *Generalized Functions*, vol. 1, Academic.

HASEGAWA, A. 1975 Dynamics of an ensemble of plane waves in nonlinear dispersive media. *Phys. Fluids* **18**, 77–79.

HASSELMANN, K. 1962 On the non-linear energy transfer in a gravity-wave spectrum. Part 1. General theory. *J. Fluid Mech.* **12**, 481.

HASSELMANN, K. 1963 On the non-linear energy transfer in a gravity-wave spectrum. Part 2. Conservation theorems, wave-particle analogy, irreversibility. *J. Fluid Mech.* **15**, 273.

HASSELMANN, K., BARNETT, T. P., BOUWS, E., CARLSON, H., CARTWRIGHT, D. E., ENKE, K., EWING, J. A., GIENAPP, H., HASSELMANN, D. E., KRUSEMAN, P., MEERBURG, A., MÜLLER, P., OLBERS, D. J., RICHTER, K., SELL, W. & WALDEN, H. 1973 Measurements of wind-wave growth and swell decay during the Joint North Sea Wave Project (JONSWAP). *Deutsche Hydrogr. Z., Suppl.* **A(80)**, no. 12.

INFELD, E. 1981 Quantitative theory of the Fermi–Pasta–Ulam recurrence in the nonlinear Schrödinger equation. *Phys. Rev. Lett.* **47**, 717–718.

JANSSEN, P. A. E. M. 1981 Modulational instability and the Fermi–Pasta–Ulam recurrence. *Phys. Fluids* **24**, 23–26.

JANSSEN, P. A. E. M. & RASMUSSEN, J. 1981 Limit cycle behavior of the bump-on-tail instability. *Phys. Fluids* **24**, 268–273.

KAMPEN, N. G. VAN 1955 On the theory of stationary waves in plasmas. *Physica* **21**, 949–963.

LAKE, B. M., YUEN, H. C., RUNGALDIER, H. & FERGUSON, W. E. 1977 Nonlinear deep-water waves: theory and experiment. Part 2. Evolution of a continuous wave train. *J. Fluid Mech.* **83**, 49–74.

LANDAU, L. 1946 On the vibrations of the electronic plasma. *J. Phys.* (USSR) **10**, 25–34.

LONGUET-HIGGINS, M. S. 1976 On the nonlinear transfer of energy in the peak of a gravity-wave spectrum: a simplified model. *Proc. R. Soc. Lond.* **A347**, 311–328.

NAYFEH, A. H. & SARIC, W. C. 1972 Nonlinear waves in a Kelvin–Helmholtz flow. *J. Fluid Mech.* **55**, 311.

PHILLIPS, O. M. 1960 On the dynamics of unsteady gravity waves of finite amplitude. Part 1. *J. Fluid Mech.* **9**, 193–217.

PHILLIPS, O. M. 1977 *The Dynamics of the Upper Ocean*, 2nd edn. Cambridge University Press.

SIMON, A. 1968 Nonlinear transport in weakly ionized plasma. *Phys. Fluids* **11**, 1181.

SIMON, A. & ROSENBLUTH, M. N. 1976 Single-mode saturation of the bump-on-tail instability: immobile ions. *Phys. Fluids* **19**, 1567–1580.

TAPPERT, F. D. 1971 Derivation of the collisionless wave kinetic equation. *SIAM Rev.* **13**, 281–282.

WATSON, K. M. & WEST, B. J. 1975 A transport equation description of nonlinear ocean surface wave interactions. *J. Fluid Mech.* **70**, 815–826.

WIGNER, E. 1932 On the quantum correction for thermodynamic equilibrium. *Phys. Rev.* **40**, 749–759.

WILLEBRAND, J. 1975 Transport in a nonlinear and inhomogeneous random gravity wave field. *J. Fluid Mech.* **70**, 113–126.

YUEN, H. C. & FERGUSON, W. E. 1978 Relationship between Benjamin–Feir instability and recurrence in the nonlinear Schrödinger equation. *Phys. Fluids* **21**, 1275–1278.

ZAKHAROV, V. E. 1968 Stability of periodic waves of finite amplitude on the surface of a deep fluid. *Zh. Prikl. Mekh. Tekh. Fiz.* **9**, 86–94. [English transl. in *J. Appl. Mech. Tech. Phys.* **9**, 190–194.]